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LETTER TO THE EDITOR

Berry's phase for supercoherent states

H A Schmitt† and A Mufti‡

† Naval Warfare Assessment Center, Pomona, CA 91769, USA
‡ Department of Physics and Arizona Materials Laboratory, University of Arizona, Tucson, AZ 85721, USA

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Abstract. Berry's phase is calculated for coherent states of the supersymmetric harmonic oscillator with the parameters of the coherent states taken to be slowly changing. Interestingly, these supercoherent states are eigenstates of the displaced Jaynes–Cummings Hamiltonian.

The adiabatic theorem of quantum mechanics states that a system prepared in an eigenstate of its Hamiltonian will remain in an instantaneous eigenstate, as the Hamiltonian is varied, provided that the variation is carried out slowly enough [1]. If the motion is cyclic, then the system will return to its original eigenstate multiplied by a dynamical phase factor [1]. In 1984, Berry [2] showed that this theorem is incomplete. In addition to the dynamical phase, the system will also acquire a geometrical phase. This phase is non-integrable and depends on the path transversed in parameter space. Berry proposed that this phase should be observable by interfering two properly prepared systems.

Much experimental and theoretical work on geometrical phases has appeared since Berry's seminal work. The theoretical work can be roughly classified into three areas:

(i) generalising the conditions under which such geometrical phases should exist;

(ii) providing alternative methods for calculating the phase; and

(iii) reinterpreting other quantal phases as manifestations of the more general geometrical phase.

The majority of the progress in area (ii) has come from group theoretical methods [3], WKB [4], coherent states [5], and path integral formalisms [6]. Examples of work in area (iii) are Bohr–Sommerfeld and Maslov [7], Aharanov–Bohm [8] and Born–Oppenheimer [9] phases. While areas (ii) and (iii) are of great interest theoretically, perhaps the most important area is (i).

As originally formulated, Berry's argument required that the eigenstate be non-degenerate and that the evolution be adiabatic and cyclic. All three requirements have now been removed [10], thus allowing a much wider range of applicability. Experimentally, geometrical phases have been observed in optical fibres [11], laser inteferometry [12], spin resonance [13], molecular physics [14] and neutron spin [15]. More recently, the time evolution of Berry's phase has been studied [16].

Coherent states have long held interest as providing a method of examining the semiclassical behaviour of systems [17]. Much of the renewed interest in coherent states has arisen from the identification of squeezed states as generalised coherent states associated with SU(1, 1) and SU(2) [18], and for obtaining information on the

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topology of the system. A path integral formalism has been used to explit this for the coherent states of the groups SU(2), SU(1, 1) and others [9].

In this letter, we study the supercoherent states introduced by Aragone and Zypman [20] and further studied by Orszag and Salamo [21]. These supercoherent states are eigenstates of strong coupling limits of the displaced Jaynes-Cummings Hamiltonian.

Supercoherent states were introduced in [20] as eigenstates of the supersymmetric annihilation operator, A, of the quantum mechanical supersymmetric harmonic oscillator:

$$A=\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \qquad A^{\dagger}=\begin{bmatrix} a^{\dagger} & 0 \\ 1 & a^{\dagger} \end{bmatrix}. \tag{1}$$

There are two linearly independent orthogonal supercoherent states, which are referred to in [20] as fermionic and supersymmetric. The fermionic supercoherent state is a minimum uncertainity state in the sense of Perelomov [22]. Explicitly,

$$|z_f\rangle=\begin{bmatrix} |z\rangle \\ 0 \end{bmatrix} \qquad |z_s\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix} z^*|z>-a^{\dagger}|z\rangle \\ |z\rangle \end{bmatrix} \tag{2}$$

where $|z\rangle=\mathscr{D}(z)|0\rangle$. In (3), $\mathscr{D}(z)=\exp(za^{\dagger}-z^*a)$, the displacement operator for the usual harmonic oscillator coherent state.

In [21] it was shown that the supercoherent state can be generated by a displacement operator, which is diagonal for the fermionic supercoherent state:

$$\mathscr{D}(z)=\begin{bmatrix} \mathscr{D}(z) & 0 \\ 0 & \mathscr{D}(z) \end{bmatrix}. \tag{3}$$

We now ask the question 'for what Hamiltonian are these supercoherent states eigenstates?' Consider the operator

$$\mathscr{H}(0)=\omega A^{\dagger}A=\omega\begin{bmatrix} a^{\dagger}a & a \\ a^{\dagger} & 1+a^{\dagger}a \end{bmatrix}. \tag{4}$$

Upon substitution, we find

$$\begin{aligned}\mathscr{H}(t)&=\omega\mathscr{D}(z)A^{\dagger}A\mathscr{D}(z)^{\dagger}\\ &=\omega\begin{bmatrix} \mathscr{D}(z) & 0 \\ 0 & \mathscr{D}(z) \end{bmatrix}\begin{bmatrix} a^{\dagger}a & a \\ a^{\dagger} & 1+a^{\dagger}a \end{bmatrix}\begin{bmatrix} \mathscr{D}(z)^{\dagger} & 0 \\ 0 & \mathscr{D}(z)^{\dagger} \end{bmatrix}\\ &=\omega\begin{bmatrix} \mathscr{D}(z)a^{\dagger}a\mathscr{D}(z)^{\dagger} & \mathscr{D}(z)a\mathscr{D}(z)^{\dagger} \\ \mathscr{D}(z)a^{\dagger}\mathscr{D}(z)^{\dagger} & 1+\mathscr{D}(z)a^{\dagger}a\mathscr{D}(z)^{\dagger} \end{bmatrix}.\end{aligned} \tag{5}$$

Direct calculation of (5) yields

$$\mathscr{H}(t)=H_{\mathrm{B}}^{D}1+\tfrac{1}{2}\omega\sigma_3+\omega(a^{\dagger}-z^*)\sigma_-+\omega(a-z)\sigma_+ \tag{6a}$$

with

$$H_{\mathrm{B}}^{D}=\omega(a^{\dagger}-z^*)(a-z)+\tfrac{1}{2}\omega \tag{6b}$$

and

$$\sigma_3=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_+=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \sigma_-=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{6c}$$

These are not the usual Pauli matrices, but it is simple to check that they form a realisation of su(2). The notation in (5) is introduced in anticipation of the construction of Berry's phase. Comparison of (6) with [23] shows that this Hamiltonian is just the strong coupling limit of the displaced Jaynes-Cummings Hamiltonian. We can also rewrite (6*a*) as:

$$\mathscr{H}(t) = H_B^D + \boldsymbol{a}\cdot\boldsymbol{\sigma} = \mathscr{D}(z)a^{\dagger}\mathscr{H}^{JC}\mathscr{D}(z)^{\dagger} \tag{7a}$$

with

$$\boldsymbol{a} = \tfrac{1}{2}\omega[(a^{\dagger}-z^*)+(a-z), \mathrm{i}(a^{\dagger}-z^*)-\mathrm{i}(a-z), 1] \tag{7b}$$

and

$$\mathscr{H}^{JC} = \omega(a^{\dagger}a+\tfrac{1}{2})+\tfrac{1}{2}\omega\sigma_3+\omega a^{\dagger}\sigma_- + \omega a\sigma_+ . \tag{7c}$$

To formulate a transition amplitude for the supercoherent states we consider a Hamiltonian constructed from the generators of the group. The propagator is then given by the time-ordered exponential:

$$T = \mathscr{T}\left\langle z_2 \left| \exp\left(-\frac{\mathrm{i}}{\hbar}\int_t^{t'} H(t)\,\mathrm{d}t\right)\right| z_1\right\rangle. \tag{8}$$

In (8), the supercoherent state is taken to be either a fermionic or a supersymmetric supercoherent state. As usual, we divide the time interval $\Delta T = t_2 - t_1$ into n equal parts $\varepsilon = (t'-t)/n$ and take the limit as $n\to\infty$. Thus to first order in ε, we obtain:

$$T = \prod_k \left\langle z' \left| \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar} H(k)\right)\right|\right\rangle \tag{9}$$

where $H(k) = H(t_k)$. The decomposition of unity is inserted between each of the equal time intervals, leading to:

$$\begin{aligned} T &= \lim_{n\to\infty}\int\ldots\int \prod_k \mathrm{d}\mu(z_k)\prod_k \left\langle z_{k+1}\left|\left(1-\frac{\mathrm{i}\varepsilon}{\hbar}H(k)\right)\right| z_k\right\rangle \\ &= \lim_{n\to\infty}\int\ldots\int \prod_k \mathrm{d}\mu(z_k)\prod_k \langle z_{k+1}|z_k\rangle\left(1-\frac{\mathrm{i}\varepsilon}{\hbar}\mathscr{H}(k)\right) \\ &= \lim_{n\to\infty}\int\ldots\int \prod_k \mathrm{d}\mu(z_k)\prod_k \langle_{z+1}|z_k\rangle \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}\mathscr{H}(k)\right) \end{aligned} \tag{10}$$

where $\mathscr{H}(k) = \langle z_k|H(k)|z_k\rangle$. The notation here is that $t_2 = t_n$ and $t_1 = t_0$. The overlap between supercoherent states is given in [20]. Inserting this expression we find

$$\begin{aligned} \langle z_{k+1}|z_k\rangle &= \exp(-\tfrac{1}{2}|z_{k+1}|^2)\exp(-\tfrac{1}{2}|z_k|^2)\exp(z_{k+1}^* z_k) \\ &= \exp(-\tfrac{1}{2}(z_{k+1}^*\Delta z_{k+1} - \Delta z_{k+1}^* z_{k+1}) \end{aligned} \tag{11}$$

with $\Delta z_{k+1} = z_{k+1} - z_k$. Thus one obtains the expression for the transition amplitudes

$$T = \lim_{n\to\infty}\int\ldots\int \prod_k \mathrm{d}\mu(z_k)\prod_k \exp\left(\frac{\mathrm{i}\varepsilon}{\hbar}\left(-\frac{\hbar}{2\mathrm{i}}\left(\frac{\Delta z_{k+1}^*}{\varepsilon}z_{k+1} - \frac{\Delta z_{k+1}}{\varepsilon}z_{k+1}^*\right) - \mathscr{H}(k)\right)\right). \tag{12}$$

This can be written formally as

$$T = \int \mathscr{D}\mu(z)\exp\left(\frac{\mathrm{i}}{\hbar}\mathscr{S}\right) \tag{13a}$$

where

$$\mathcal{S}=\int_0^T \mathcal{L}\,\mathrm{d}t$$

and

$$\mathcal{L}=\frac{\hbar \mathrm{i}}{2}(z'z^*-z^{*\prime}z)-\mathcal{H} \tag{13b}$$

are the action and Lagrangian, respectively. In (14), the prime denotes time differentiation. This equation leads to Lagrange's equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial z'}-\frac{\partial\mathcal{L}}{\partial z}=0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial z^{*\prime}}-\frac{\partial\mathcal{L}}{\partial z^*}=0 \tag{14}$$

from which we obtain

$$z'=-\frac{\mathrm{i}}{\hbar}\frac{\partial\mathcal{H}}{\partial z^*} \qquad z^{*\prime}=\frac{\mathrm{i}}{\hbar}\frac{\partial\mathcal{H}}{\partial z}. \tag{15}$$

We now consider the strong coupling limit of the displaced Jaynes-Cummings Hamiltonian given in ($7c$) and the fermionic supercoherent states. We can easily show that the supercoherent states are eigenstates of this Hamiltonian. In fact, by following the procedure in [5], we want to construct a time-dependent Hamiltonian of the form $H(t)=U(t)H(0)U^{\dagger}$. The Berry's phase is then constructed by taking $U(t)=\mathcal{D}(g(t))$, where g is a generator in the Lie algebra, and $H(0)$ is restricted to the Cartan subalgebra. Then, starting with an eigenstate of $H(0)$, $|\alpha\rangle$, the state at time t is a generalised coherent state $\mathcal{D}(g(t))|\alpha\rangle$ [22].

By taking the parameters of the coherent state to be slowly varying, we can construct the Berry phase for the system. By differentiation of the supercoherent state or by use of the path integral formalism, we find:

$$\dot{\gamma}=\left\langle z\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|z\right\rangle=\tfrac{1}{2}(z'z^*-z^{*\prime}z). \tag{16}$$

This is the same result that one obtains for the usual harmonic oscillator coherent states. This is not entirely unexpected, since the overlaps of the supercoherent states and that for the usual harmonic oscillator coherent states are the same.

We have used a path integral formalism to arrive at the equations of motion and Berry's phase for the supercoherent states introduce in [20]. Following the well defined prescription in [5], we have constructed our time-dependent Hamiltonian, which is the strong coupling limit of the Jaynes-Cummings Hamiltonian. This Hamiltonian has been studied recently in connection with the generation of second harmonics and as well as with the interaction of one atom with a single mode field.

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